International Journal of Theoretical Physics, Vol. 44, No. 3, March 2005 (© 2005)
DOI: 10.1007/s10773-005-3645-x

Inverse Variational Problem for Nonlinear Evolution Equations

S. Ghosh,[1] U. Das,[2] and B. Talukdar[3]

Received December 4, 2004; accepted February 22, 2005

The Helmholtz solution of the inverse problem for the variational calculus is used to study the analytic or Lagrangian structure of a number of nonlinear evolution equations. The quasilinear equations in the KdV hierarchy constitute a Lagrangian system. On the other hand, evolution equations with nonlinear dispersive terms (FNE) are non-Lagrangian. However, the method of Helmholtz can be judiciously exploited to construct Lagrangian system of such equations. In all cases the derived Lagrangians are gauge equivalent to those obtained earlier by the use of Hamilton's variational principle supplemented by the methodology of integer-programming problem. The free Hamiltonian densities associated with the so-called gauge equivalent Lagrangians yield the equation of motion *via* a new canonical equation similar to that of Zakharov, Faddeev and Gardner. It is demonstrated that the Lagrangian system of FNE equations supports compacton solutions.

KEY WORDS: evolution equations; fully nonlinear; compacton solutions.
PACS: 47.20.Ky; 42.81.Dp

1. INTRODUCTION

The calculus of variations plays a crucial role in a wide variety of physical problems ranging from classical mechanics to quantum field theory. Determination of the time behavior of a mechanical system is a typical variational problem of classical dynamics. Here one deals with a special class of functionals called the action. The process of minimizing the action functional for the variation of a function goes by the name *Hamilton's variational principle*. Euler first discovered the necessary condition that a minimizing function must satisfy. Now a days this is known as the Euler-Lagrange equation and the function which satisfies this equation is called the Lagrangian function. It is widely believed that all physical information of a system is encoded in the Lagrangian function. For complex

[1] Patha-Bhavana, Visva-Bharati University, Santiniketan 731235 India.
[2] Abhedananda Mahavidyalaya, Sainthia 731234, India.
[3] Department of Physics, Visva-Bharati University, Santiniketan 731235, India; e-mail: binoy123@sancharnet.in.

dynamical systems, the method of Euler and Lagrange is more powerful than Newton's laws of motion. Towards the end of nineteenth century, the concept of differentiation was generalized by Volterra, Hadamard and three of his students Fréchet, Gâteaux and Hilbert to infinite dimensional spaces thereby providing variational calculus with a solid base for applications to mechanics of continuous media and nonlinear evolution equations (Blanchard and Br*ü*ning, 1992). In the calculus of variations one is concerned with two types of problems, namely, the direct and inverse problem of newtonian mechanics. The direct problem is essentially the conventional one in which one first assigns a Lagrangian and then computes the equations of motion through Lagrange's equations. As opposed to this, the inverse problems begins with the equations of motion and then constructs a Lagrangian consistent with the variational principle.

In the present work we shall deal with the inverse problem of variational calculus for the quasilinear equations of the Korteweg-de Vries (KdV) hierarchy (Lax, 1968) as well as a fully nonlinear evolution (FNE) equation (Rosenau and Hyman, 1993). We call equations of the KdV hierarchy as quasilinear because the dispersive behavior of the solution of each equation is governed by a linear term. The dispersion produced is compensated by nonlinear effects resulting in the formation of exponentially localized solitons. As opposed to this, the dispersive term is nonlinear for a FNE equation. In general, the solitary wave solutions of FNE equations compactify under nonlinear dispersion to produce deep qualitative changes in the nature of genuinely nonlinear phenomena leading to the formation of compactons, cuspons, and tipons (Rosenau, 1997).

The inverse problem of the calculus of variation was solved by Helmholtz during the end of the nineteenth century (Olver, 1993). For continuum mechanics the Helmholtz version of the inverse problem proceeds by considering an $r-$tuple of differentiable functions written as

$$P[u] = P\left(x,\, u^{(n)}\right) \epsilon \mathcal{A}^r \tag{1}$$

and then defining the so-called Fréchet derivative. The Fréchet derivative of P is the differential operator $D_P; \mathcal{A}^q \rightarrow \mathcal{A}^r$ and is given by

$$D_P(Q) = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} P\left[u + \epsilon\, Q[u]\right] \tag{2}$$

for any $Q\epsilon\mathcal{A}^q$. The Helmholtz condition asserts that P is the Euler-Lagrange expression for some variational problem iff D_P is self-adjoint. When self-adjointness is guaranteed, a Lagrangian for P can be explicitly constructed using the homotopy formula

$$\mathcal{L}[u] = \int_0^1 u\, P\left[\lambda u\right]\, d\lambda. \tag{3}$$

The solution presented above is very neat, from the wider perspective of determining which systems of differential equations arise from variational principle. However, there are situations where the Helmholtz solution may turn out to be somewhat unsatisfactory. For example, if D_P is not self-adjoint for a system of differential equations, we have to stop by inferring that the system is non-Lagrangian. But one may be interested to construct similar system of equations, which follows from Lagrangians because a non-Lagrangian system does not allow one to carry out a linear stability check (Dey and Khare, 2000) as well as to derive a field theory (Rosenau and Hyman, 1993).

In the present work we show that KdV and higher KdV equations satisfy Helmholtz condition and form a Lagrangian system. As opposed to this, the FNE equations are non-Lagrangian. We directly use the Helmholtz solution of the inverse problem to construct expressions for Lagrangians for the equations in the KdV hierarchy. We adapt the Helmholtz solution to introduce a Lagrangian system of FNE equations. We observe that in both cases our solution of the inverse problem leads to Lagrangian densities which are gauge equivalent to those obtained by us using a method of dimensional analysis (Talukdar *et al.*, 2002, 2003). Moreover, the free Hamiltonian densities obtained from the so-called gauge equivalent Lagrangians do not yield the equation of motion when substituted in the canonical equation of Zakharov and Faddeev (1971) and of Gardner (1971). This observation has an old root in classical mechanics literature (Currie and Saletan, 1966). We take this opportunity to introduce another canonical equation which is consistent with these Hamiltonian densities. In Sec. 2 we deal with the equations of the KdV hierarchy and find that Hamiltonian densities obtained from our Lagrangians satisfy a new canonical equation. In Section 3 we carry out a similar analysis for the fully nonlinear evolution equation of Rosenau and Hyman (1993) and find that this equation is non-Lagrangian. However, demanding variational self-adjointness of a suitably chosen Euler-Lagrange expression ($P[u]$), we could introduce a Lagrangian system of FNE equations which exhibit the same canonical structure as observed for the KdV equations. Here we also demonstrate that the modified Rosenau-Hyman equation having a Lagrangian structure possesses compacton solution. Finally, we make some concluding remarks in Section 4.

2. EQUATIONS OF THE KDV HIERARCHY

One common trick to put the KdV equation

$$u_t = u_{3x} + 6uu_x \tag{4}$$

into the variational form is to replace $u(x, t)$ by a potential function defined by

$$w(x, t) = \int_x^\infty dy\, u(y, t) \tag{5}$$

such that

$$w_x(x,\ t) = -u(x,\ t). \tag{6}$$

Note that this trick works only for equations which are of odd order in space derivatives and, fortunately for us, all higher KdV equations belong to this class. From (4) and (6), we get

$$w_{xt} = w_{4x} - 6w_x w_{2x} \tag{7}$$

which can be integrated from $-\infty$ to x to give

$$w_t = w_{3x} - 3w_x^2. \tag{8}$$

Here we have applied the boundary condition, $u(-\infty,\ t) \to 0$. Calogero (1982) could reduce (8) to the variational form

$$\delta \int_{t_1}^{t_2} dt \int_{-\infty}^{+\infty} dx\, \mathcal{L}(w_t,\ w_x,\ w_{2x}) = 0 \tag{9}$$

to define Lagrangian density

$$\mathcal{L}_v = \frac{1}{2} w_t w_x + \frac{1}{2} w_{2x}^2 + w_x^3\,. \tag{10}$$

Let us now check if the result in (10) also follows from the Helmholtz theorem. To achieve this we write from (7) the Euler-Lagrange expression

$$P[w] = w_{4x} - 6w_x w_{2x}\,. \tag{11}$$

Using (11), (2) gives the self-adjoint differential operator

$$D_P = D_{4x} - 6w_x D_{2x} - 6w_{2x} D_x, \quad D_{nx} = \left(\frac{d}{dx}\right)^n \tag{12}$$

proving the existence of the Lagrangian for (4) or (7). The time independent part of the Lagrangian density for $P[w]$ can be obtained from (3) to write

$$\mathcal{L}_2 = \frac{1}{2} w w_{4x} - 2w w_x w_{2x}\,. \tag{13}$$

The total Lagrangian density for (7) is obtained by adding a velocity (w_t) dependent part, $\mathcal{L}_1 = \frac{1}{2} w_t w_x$ to $\mathcal{L}_2$ and we have

$$\mathcal{L}_{\text{h}} = \frac{1}{2} w_t w_x + \frac{1}{2} w w_{4x} - 2w w_x w_{2x}\,. \tag{14}$$

The subscripts v and h of $\mathcal{L}$ in (10) and (14) merely indicate that these results have been obtained by direct application of variational principle and by making use of the homotopy formula. Both Lagrangians are of first order in time derivative but they differ in the order of space derivatives. In particular, $\mathcal{L}_v$ is of second order and $\mathcal{L}_h$ is of fourth order. It is interesting to note that both $\mathcal{L}_v$ and $\mathcal{L}_h$ lead to the same canonical momentum density

$$\pi = \frac{1}{2} w_x. \tag{15}$$

This equation can not be inverted for the velocity w_t implying that the Lagrangian densities are degenerate (Sudarshan and Mukanda, 1974). Therefore, one must use the Dirac's theory of constraints (Dirac, 1964) to obtain the total Hamiltonian density given by

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1. \tag{16}$$

Here $\mathcal{H}_0$ is the free part of $\mathcal{H}$ determined by the usual Legendre map and evaluation of the expression for $\mathcal{H}_1$ requires the explicit use of Dirac's theory. The Hamiltonian densities $\mathcal{H}_{0v}$ and $\mathcal{H}_{0h}$ corresponding to $\mathcal{L}_v$ and $\mathcal{L}_h$ can be obtained as

$$\mathcal{H}_{0v} = -\frac{1}{2} w_{2x}^2 - w_x^3 \tag{17}$$

and

$$\mathcal{H}_{0h} = -\frac{1}{2} w w_{4x} + 2 w w_x w_{2x}. \tag{18}$$

More than three decades ago Zakharov and Faddeev (1971) and Gardner (1971) interpreted the KdV equation as a completely integrable Hamiltonian system in an infinite dimensional phase space. The Hamiltonian form of (4) is given by

$$u_t = \partial_x \frac{\delta \mathcal{H}}{\delta u} \tag{19}$$

with $\partial_x = \frac{\partial}{\partial x}$, the Hamiltonian operator and $\mathcal{H}$, the free Hamiltonian density. From (6) and (19) we have

$$w_t = \frac{\delta \mathcal{H}}{\delta w_x}. \tag{20}$$

Here the variational derivative

$$\frac{\delta}{\delta v} = \sum_n (-\partial_x)^n \frac{\partial}{\partial v_n}, \quad v_n = (\partial_x)^n v. \tag{21}$$

Interestingly, we find that (17) and (20) give the KdV equation as written in (8) while (18) and (20) do not lead to the same equation. This tends to imply that $\mathcal{H}_{0h}$ obtained from $\mathcal{L}_h$ is inconsistent with the canonical equation of Faddeev, Zakharov and Gardner. Thus one would like to check if $\mathcal{L}_v$ and $\mathcal{L}_h$ stand for analytic representations (Santili, 1984) of the same equation. Interestingly, both Lagrangians when used in the appropriate Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial w_t}\right) - \frac{\delta \mathcal{L}}{\delta w} = 0 \tag{22}$$

give the same equation of motion as given in (8). Thus these two Lagrangians must be gauge equivalent. We find that

$$\mathcal{L}_h = \mathcal{L}_v + \partial_x \left(\frac{1}{2} w w_{3x} - \frac{1}{2} w_x w_{2x} - w w_x^2\right). \tag{23}$$

Therefore, one could reasonably ask: why are (18) and (20) inconsistent? To get a plausible answer to this question we recall the celebrated work of Currie and Saletan (1966) who observed that Hamiltonians obtained from gauge equivalent Lagrangians correspond to different canonical structure. Keeping this in view, we venture to suggest a new canonical equation

$$w_{xt} = -\frac{\delta \mathcal{H}}{\delta w} \tag{24}$$

as a supplement of (20). This equation gives (7) or (8) for the Hamiltonian $\mathcal{H}_{0h}$ obtained by using the Helmholtz theorem. In the following we demonstrate that (24) holds good for any member of the KdV hierarchy.

For higher order equations the simple-minded approach used in Calogero (1982) does not work and one takes recourse to other methods like the dimensional analysis (Talukdar *et al.*, 2002, 2003). However, it can easily be verified that the Fréchet derivative of every member of the KdV hierarchy is self-adjoint and (3) can be used to derive an expression for the Lagrangian. For the second and third members of the hierarchy

$$u_t = u_{5x} - 10 u u_{3x} - 20 u_x u_{2x} + 30 u^2 u_x \tag{25}$$

and

$$\begin{aligned} u_t = {} & u_{7x} + 14 u u_{5x} + 42 u_x u_{4x} + 70 u_{2x} u_{3x} + 70 u^2 u_{3x} \\ & + 280 u u_x u_{2x} + 70 u_x^3 + 140 u^3 u_x \end{aligned} \tag{26}$$

we have

$$\mathcal{L}_h^{(5)} = \frac{1}{2} w_t w_x + \frac{1}{2} w w_{6x} + \frac{10}{3} w w_x w_{4x} + \frac{20}{3} w w_{2x} w_{3x} + \frac{30}{4} w w_x^2 w_{2x} \tag{27}$$

and

$$\begin{aligned} \mathcal{L}_h^{(7)} = {} & \frac{1}{2} w_t w_x + \frac{1}{2} w w_{8x} - 14 w w_{2x} w_{5x} - \frac{14}{3} w w_x w_{6x} - \frac{70}{3} w w_{3x} w_{4x} \\ & + 70 w w_x w_{2x} w_{3x} + \frac{35}{2} w w_x^2 w_{4x} + \frac{35}{2} w w_{2x}^3 - 28 w w_x^3 w_{2x}. \end{aligned} \tag{28}$$

Here the superscripts 5 and 7 merely indicate that the results in (27) and (28) refer to fifth and seventh order equations. The free part of the Hamiltonian density corresponding to (27) and (28) are given by

$$\mathcal{H}_{0h}^{(5)} = -\frac{1}{2}ww_{6x} - \frac{10}{3}ww_xw_{4x} - \frac{20}{3}ww_{2x}w_{3x} - \frac{30}{4}ww_x^2w_{2x} \tag{29}$$

and

$$\begin{aligned}\mathcal{H}_{0h}^{(7)} = & -\frac{1}{2}ww_{8x} + 14ww_{2x}w_{5x} + \frac{14}{3}ww_xw_{6x} + \frac{70}{3}ww_{3x}w_{4x} \\ & - 70ww_xw_{2x}w_{3x} - \frac{35}{2}ww_x^2w_{4x} - \frac{35}{2}ww_{2x}^3 + 28ww_x^3w_{2x}.\end{aligned} \tag{30}$$

One can easily verify that (29) and (30) when used in (24) give (25) and (26) written in terms of w.

Some remarks on the canonical Equation (24) are now in order. Following Currie and Saletan (1966) let us construct a gauge equivalent Lagrangian for $\mathcal{L}_v$ in (10). To that end we choose the gauge term so as to cancel the second and third terms of $\mathcal{L}_v$. Obviously, the gauge term will be a sum of bilinear combination of w, w_x and w_{2x}, that have the correct dimension of the terms in (10). Thus we write

$$\mathcal{L}_v^{(g)} = \frac{1}{2}w_tw_x - \frac{1}{2}w_{2x}^2 + w_x^3 - \frac{d}{dx}\left(a_1ww_{3x} + a_2w_xw_{2x} + a_3ww_x^2\right) \tag{31}$$

with a_i's as arbitrary constants. For the desired cancellation we must have $a_1 = 0$, $a_2 = -\frac{1}{2}$ and $a_3 = 1$ giving a third-order Lagrangian for the KdV equation

$$\mathcal{L}_v^{(g)} = \frac{1}{2}w_tw_x + \frac{1}{2}w_xw_{3x} - 2ww_xw_{2x}. \tag{32}$$

The Lagrangian in (32) is different from $\mathcal{L}_\mathrm{h}$ in (14). Despite that the Hamiltonian density constructed from (32) *via* (24) gives the KdV Equation in (7) or (8). In fact, one can verify that all gauge equivalent Lagrangians corresponding to those obtained by direct use of Hamilton's variational principle (Talukdar *et al.*, 2002, 2003; Calogero, 1982) lead to Hamiltonian densities which satisfy the canonical Equation (24) and not the original equation written by Zakharov, Faddeev and Gardner.

3. EQUATIONS WITH NONLINEAR DISPERSIVE TERMS

One of the FNE equations introduced by Rosenau and Hyman (1993) is given by

$$u_t + 3u^2u_x + 6u_xu_{2x} + 2uu_{3x} = 0. \tag{33}$$

We begin this section by examining if the condition of self-adjointness holds good for this equation. As in the case of KdV hierarchy it will be convenient to work with $w(x, t)$ rather than $u(x, t)$. From (33), the differential equation for $w(x, t)$ is obtained in the form

$$w_{xt} = -3w_x^2 w_{2x} + 6w_{2x}w_{3x} + 2w_x w_{4x}. \tag{34}$$

Equivalently,

$$w_t + w_x^3 - 2w_{2x}^2 - 2w_x w_{3x} = 0. \tag{35}$$

From (34) we write the Euler-Lagrange expression as

$$P[w] = -3w_x^2 w_{2x} + 6w_{2x}w_{3x} + 2w_x w_{4x}. \tag{36}$$

From (2) and (36)

$$D_P = 6w_x w_{2x} D_x + 3w_x^2 D_x^2 - 6w_{2x} D_x^3 - 6w_{3x} D_x^2 - 2w_x D_x^4 - 2w_{4x} D_x. \tag{37}$$

To construct the adjoint operator D_P^* of the above Fréchet derivative we rewrite (37) as

$$D_P = \sum_{j=1}^{4} P_j(w) D_j \tag{38}$$

and make use of the definition (Olver, 1993)

$$D_P^* = \sum_{j=1}^{4} (-D_j) P_j(w). \tag{39}$$

This gives

$$D_P^* = 6w_x w_{2x} D_x + 3w_x^2 D_x^2 - 2w_{2x} D_x^3 - 2w_x D_x^4. \tag{40}$$

Clearly, $D_P \neq D_P^*$ verifying that the Fréchet derivative of $P[w]$ in (36) is non-self-adjoint. Thus (33) does not have an analytic representation (Santili, 1984) to follow from a Lagrangian density and the variational structure of the system remains undiscovered. This represents an awkward analytical constraint for the equation of Rosenau and Hyman. A non-Lagrangian system does not allow one to carry out a linear stability check as well as derive a field theory for particles described by compactons. In the recent past, two of us made use of a heuristic approach (Talukdar *et al.*, 2003) to introduce Lagrangian system of FNE equations. We now show that same objective can be achieved within the framework of the Helmholtz theorem. We rewrite the Euler-Lagrange expression for $P[w]$ in (36) as

$$P[w] = \alpha w_x^2 w_{2x} + \beta w_{2x} w_{3x} + \gamma w_x w_{4x} \tag{41}$$

and demand that there exists a choice for the constants α, β and γ such that $P[w]$ in (41) is self-adjoint. The Fréchet derivative for (41) can easily be obtained as

$$D_P = 2\alpha w_x w_{2x} D_x + \alpha w_x^2 D_{2x} + \beta w_{2x} D_{3x} + \beta w_{3x} D_{2x} + \gamma w_x D_{4x} + \gamma w_{4x} D_x. \tag{42}$$

We now make use of (39) to write a self-adjoint operator corresponding to (42) and find

$$\begin{aligned} D_P^* &= 2\alpha w_x w_{2x} D_x + \alpha w_x^2 D_{2x} + (4\gamma - \beta) w_{2x} D_{3x} \\ &\quad + 2(3\gamma - \beta) w_{3x} D_{2x} + \gamma w_x D_{4x} + (3\gamma - \beta) w_{4x} D_x. \end{aligned} \tag{43}$$

From our demand of the variational self-adjointness we obtain from (42) and (43) a relation between β and γ as

$$\gamma = \frac{1}{2}\beta \tag{44}$$

while α remains totally arbitrary. Thus we would expect that

$$w_{xt} = \alpha w_x^2 w_{2x} + \beta w_{2x} w_{3x} + \frac{\beta}{2} w_x w_{4x} \tag{45}$$

stands for a family of FNE equations for any choice of α and β. The special result $\alpha = 3$, $\beta = 12$, was obtained by two of us (Talukdar *et al.*, 2003) in a relatively recent publication. The Lagrangian density for (45) obtained from the homotophy formula (3) is given by

$$\mathcal{L}_h = \frac{1}{2} w_t w_x + \frac{\alpha}{4} w w_x^2 w_{2x} + \frac{\beta}{3} w w_{2x} w_{3x} + \frac{\beta}{6} w w_x w_{4x}. \tag{46}$$

The free particle Hamiltonian density for (46) is written as

$$\mathcal{H}_{0h} = -\frac{\alpha}{4} w w_x^2 w_{2x} - \frac{\beta}{3} w w_{2x} w_{3x} - \frac{\beta}{6} w w_x w_{4x}. \tag{47}$$

Equation (47) can be used in (24) to get (45). This serves as useful check on the canonical equation introduced by us. It remains an interesting curiosity to examine if (45) supports compacton solutions. In the following we deal with this.

In terms of $u(x, t)$ (45) reads

$$u_t + \beta u_x u_{2x} + \frac{\beta}{2} u u_{3x} - \alpha u^2 u_x = 0. \tag{48}$$

Let us assume a solution of (48) in the form of a traveling wave

$$u(x, t) = f(x + \lambda t) \equiv f(z), \tag{49}$$

with λ, the velocity of propagation of $u(x, t)$. From (48) and (49) we get the ordinary differential equation

$$\lambda \frac{df}{dz} + \beta \frac{df}{dz} \frac{d^2 f}{dz^2} + \frac{\beta}{2} f \frac{d^3 f}{dz^3} - \alpha f^2 \frac{df}{dz} = 0. \tag{50}$$

Imposing appropriate boundary conditions (50) can be integrated twice to get

$$\left(\frac{df}{dz}\right)^2 = \frac{f}{\beta}\left(\frac{\alpha}{3} f^2 - 2\lambda\right). \tag{51}$$

We have verified that for (48) to support a compacton solution we must impose the condition $\alpha > 0$ and $\beta < 0$. This relation tends to serve as a constraint on the choice of α and β. For $\alpha = 3$ and $\beta = -1$, (51) can be solved to get (Wolfram, 2000)

$$f = \sqrt{2\lambda}\, \cos\left[2\,\text{Jacobi amplitude}\left[-\frac{\lambda^{1/4} z}{2^{3/4}},\ 2\right]\right]. \tag{52}$$

A compacton solution similar to that in (52) has also been reported by Rosenau and Hyman (1993)

4. CONCLUSION

The Hamiltonian structure of integrable nonlinear evolution equation is based on a mathematical formulation that does not make explicit reference to Lagrangians (Olver and Rosenau, 1996; Ghosh *et al.*, 2003). We believe that the Lagrangian approach is quite interesting because here one can derive all physico-mathematical results from first principles. In this work, we have seen that the Helmholtz solution of the inverse problem for the variational calculus serves this purpose for the equations in the KdV hierarchy. However, one must reduce the order of the derived Lagrangians ($\mathcal{L}_h$) such that the corresponding Hamiltonian densities could be consistent with the canonical equation of Zakharov and Faddeev (1971) and Gardner (1971). The free Hamiltonian densities ($\mathcal{H}_{0h}$) corresponding to ($\mathcal{L}_h$) give the equations of motion *via* a new canonical equation. A similar observation in the context of particle dynamics was made by Currie and Saletan (1966) about 40 years ago.

An evolution equation for solitary waves with compact support is shown to be non-Lagrangian since the Fréchet derivative of its Euler-Lagrange expression is non-self-adjoint. We have derived a straightforward method to regain the variational self-adjointness and thus introduce a family of Lagrangian system of FNE equations. The Lagrangian and the Hamiltonian densities of the new set of equations exhibit properties similar to those found for the KdV hierarchy. About a decade ago, Cooper *et al.* (1993) introduced a family of FNE equations similar to those of us by working with an ansatz for the Lagrangian density given by

$$\mathcal{L}(l,\ p) = \frac{1}{2} w_x w_t - \frac{(w_x)^l}{l(l-1)} + \alpha (w_x)^p (w_{2x})^2. \tag{53}$$

The result in (53) appears to be a simple variant of the Lagrangian density given in Calogero (1982) and quoted by us (see (10)). However this choice is interesting since the Fréchet derivatives of the Euler-Lagrange expressions resulting from (53) are self-adjoint. Relatively recently, a generalized fifth-order KdV equation has been found to support a compacton solution (Cooper *et al.*, 2001). One can check that the variational inverse problem followed by us and the conclusion derived therefrom also apply for the fifth-order Rosenau-Hyman type equation (Rosenau, 1999).

ACKNOWLEDGMENT

One of the authors (BT) is thankful to the University Grants Commission, Government of India for a research grant (F.10-10/2003(SR)).

REFERENCES

Blanchard, Ph. and Brüning, E. (1992). *Variational Methods in Mathematical Physics*, Springer-Verlag, NY.

Calogero, F. (1982). *Spectral Transform and Solitons*, Vol I, North-Holland Pub. Co., NY.

Cooper, F., Hyman, J. M., and Khare, A. (2001). *Physical Review E* **64**, 026608.

Cooper, F., Shepard, H., and Sodano, P. (1993). *Physical Review E* **48**, 4027.

Currie, D. J. and Saletan, E. J. (1966). *Journal of Mathematical Physics* **7**, 967.

Dey, B. and Khare, A. (2000). *Journal of Physics* A **33**, 5335.

Dirac, P. A. M. (1964). *Lectures on Quantum Mechanics, Belfer Graduate School Monograph Series No. 2*, Yeshiva University, NY.

Gardner, C. S. (1971). *J. Math. Phys.* **12**, 1548.

Ghosh, S., Talukdar, B., and Shamanna, J. (2003). *Czechoslovak Journal of Physics* **53**, 425.

Lax, P. D. (1968). *Communications on Pure and Applied Mathematics* **21**, 467.

Olver, P. J. (1993). *Applications of Lie Groups to Differential Equations*, Springer-Verlag, NY.

Olver, P. J. and Rosenau, P. (1996). *Physical Review* E **53**, 1900.

Rosenau, P. (1997). *Physics Letters A* **230**, 305.

Rosenau, P. (1999). *Physics Letters A* **252**, 297.

Rosenau, P. and Hyman, J. M. (1993). *Physical Review Letters* **70**, 564.

Santili, R. M. (1984). *Foundations of Theoretical Mechanics I*, Springer-Verlag, NY.

Sudarshan, E. C. G. and Mukanda, N. (1974). *Classical Dynamics: A Modern Perspective*, John Wiley and Sons, NY.

Talukdar, B., Ghosh, S., Shamanna, J., and Sarkar, P. (2002). *European Physical Journal D* **21**, 105.

Talukdar, B., Shamanna, J., and Ghosh, S. (2003). *Pramana—Journal of Physics* **61**, 99.

Wolfram, S. (2000). *Mathematica: A system for doing mathematics by computer*, Version 4.1, Wolfram Research Inc., NY.

Zakharov, V. E. and Faddeev, L. D. (1971). *Functional Analysis and Application* **5**, 18.